THE STABILITY OF THE NORMAL AGE DISTRIBUTION[1]

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There is a unique age distribution which, in certain circumstances,[2] has the property of perpetuating itself when once set up in a population. This fact is easily established,[3] as is also the analytical form of this unique *fixed* or *normal* age distribution.

More difficult is the demonstration that this age distribution is *stable*, that a population will spontaneously revert to it after displacement therefrom, or will converge to it from an arbitrary initial age distribution. Such a demonstration has hitherto been offered only for the case of small displacements,[4] by a method making use of integral equations. The purpose of the present communication is to offer a proof of stability which employs only elementary analytical operations, and which is readily extended to cover also the case of large displacements. This method presents the further advantage that it is molded in more immediate and clearly recognizable relation to the physical causes that operate to bring about the normal age distribution.

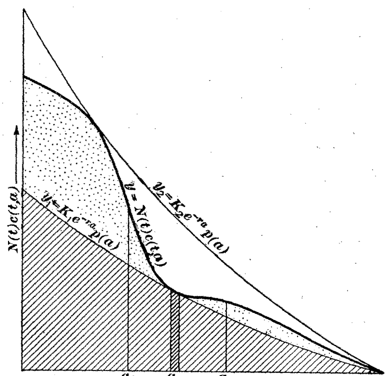

Fig. 1

Consider a population which, at time t has a given age distribution such as that represented by the heavily drawn curve in fig. 1, in which the abscissae represent ages a (in years, say), while the ordinates y are such that the area comprised between two ordinates erected at a_1 and a_2, respectively, represents the number of individuals between the ages a_1 and a_2.

If we denote by $N(t)$ the total population at time t, and if the ordinates of our curve are

$$y = N(t)\ c(t,a) \tag{1}$$

we have, evidently,

$$\int_{a_1}^{a_2} y\ da = N(t) \int_{a_1}^{a_2} c(t,a)da = N(t,a_1,a_2) \tag{2}$$

where $N(t,a_1,a_2)$ denotes the number of individuals living at time t and comprised within the age limits a_1 and a_2. We may speak of $c(t,a)$ as the coefficient of age distribution. It is, of course, in general a function of t, only in the special case of the fixed or self-perpetuating age distribution is $c(a)$ independent of the time.

Now, without assuming anything regarding the stability of the self-perpetuating age distribution, it is easy to show[2] that its form must be

$$c(a) = \frac{e^{-ra}p(a)}{\int_0^\infty e^{-ra}p(a)da} = be^{-ra}p(a) \tag{3}$$

where r is the real root of the equation

$$1 = \int_{a_i}^{a_j} e^{-ra}p(a)\beta(a)da \tag{4}$$

In this equation r is the natural rate of increase of the population, i.e., the difference $r = b - d$ between the birthrate per head b and the deathrate per head d, and $p(a)$ is the probability, at birth, that a random individual will live to age a (in other words, it is the principal function tabulated in life tables, and there commonly denoted by 1_x). The limits a_i and a_j of the integral are the lower and upper age limits of the reproductive period. The factor $\beta(a)$, which might be termed the procreation factor, or more briefly the birth factor, is the average number of births contributed per annum by a parent of age a. (In a population of mixed sexes it is, of course, immaterial, numerically, to what parent each birth is credited. It will simplify the reasoning, however, if we think of each birth as credited to the female parent only.)

The factor $\beta(a)$ will in general itself depend on the prevailing age distribution. This is most easily seen in the case of extremes, as for example in a population which should consist exclusively of males under one year of age and females over 45. But, except in such extreme cases, $\beta(a)$ will not vary greatly with changes in the age constitution of the population, and we shall first develop our argument on the supposition that $\beta(a)$ is independent of the age distribution. We shall then extend our reasoning to the more general case of $\beta(a)$ variable with $c(t,a)$.

Referring now again to fig. 1, let two auxiliary curves be drawn, a *minor tangent curve* and a *major tangent curve*

$$y_1 = K_1e^{-ra}p(a) \quad (5) \qquad y_2 = K_2e^{-ra}p(a) \quad (6)$$

the constants K_1, K_2 being so chosen that the minor tangent curve lies wholly beneath the given arbitrary curve, except where it is tangent thereto, while the major tangent curve lies wholly above the given curve, except where it is tangent thereto.

The given arbitrary curve representing the age constitution of the population at time t then lies wholly within the strip or area enclosed between the minor and the major tangent curves.

Now consider the state of affairs at some subsequent instant t'. Had the population at time t consisted solely of the individuals represented by the lightly shaded area in fig. 1, i.e. the area under the minor tangent curve, then at time t' the population would be represented by the lower curve of fig. 2, whose equation is

$$y'_1 = K'_1 e^{-ra} p(a) \qquad (7)$$

$$= K_1 e^{r(t'-t)} e^{-ra} p(a) \qquad (8)$$

For the age distribution (5) is of the fixed form (3), and therefore persists in (7); on the other hand, given such fixed age distribution, the population as a whole increases in geometric progression,[3] so that $K'_1 = K_1 e^{r(t'-t)}$. In point of fact, we have left out of reckoning that portion of the population which in fig. 1 is represented by the dotted area. Hence, in addition to the population under the lower curve of fig. 2, there will, at time t', be living a body of population which for our present purposes it is not necessary to determine numerically. We need only know that it is some positive number, so that the curve representing the actual population at time t' must lie wholly above or in contact with the curve (8).

By precisely similar reasoning it is readily shown that at time t' the actual curve lies wholly beneath or in contact with the curve

$$y'_2 = K_2 e^{r(t'-t)} e^{-ra} p(a) \qquad (9)$$

Hence at time t' the actual curve lies wholly within the strip comprised between the two curves (8) and (9).

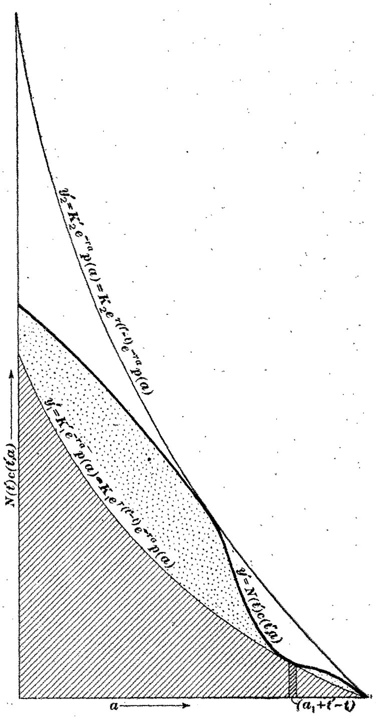

Fig. 2

The curves shown are intended to be interpreted only in a qualitative sense; so, for example, the increase in the ordinates in passing from fig. 1 to fig. 2 is very much exaggerated, to render it obvious to the eye.

Consider now an elementary strip, of width da, of the original population (shown heavily shaded in fig. 1), which at time t is in contact with the minor tangent curve. Let this con-

tact occur at age a_1, so that $y_t(a_1) = K_1 e^{-ra_1} p(a_1)$. At time t' the survivors of the individuals comprised in this elementary strip will be of age $(a_1 + t' - t)$, so that they will then be represented by a strip of width da and of altitude

$$y_{t'}(a_1 + t' - t) = K'_1 e^{-ra_1} p(a_1 + t' - t)$$
$$= K_1 e^{r(t'-t)} e^{-r(a_1 + t' - t)} p(a_1 + t' - t)$$

From this it is seen that the elementary strip of population which at time t' contacts with the minor tangent curve (8) is built up of the survivors of the strip which at time t contacted with the minor tangent curve (5). In other words, if we follow up, by identity of individuals, the element of the population which at any instant contacts with the minor tangent curve, this element (so long as any part of it survives) continues in contact with that curve. (It must be remembered, however, that the tangent curve itself changes with time according to (5), (8).)

Similarly it follows that the element of population which at any instant contacts with the major tangent curve continues in this contact so long as any part of it survives.

And again, considering any element of the population which does not contact with the minor or the major tangent curves, but has its upper extremity at some point within the area enclosed between these curves, it can be shown by precisely similar reasoning that such element continues in such intermediate position.

Turning now from the consideration of the survivors of the original population, and taking in view the new population added by births since the time $t = t$, we note that if the original population had been that represented by the shaded area in fig. 1, i.e., by the area under the minor tangent curve, then the birthrate would at all times have been such as continuously to reproduce a population represented by the minor tangent curve (5), (8).

In point of fact, provided that contact with the minor tangent curve is not *continuous* over a range of ages equal to or greater than $a_j - a_i$, it is easily seen that the total birthrate

$$B = N \int_{a_i}^{a_j} c(a)\beta(a)da$$

is always greater (equality is here excluded) than that which would result from and in turn reproduce the age distribution represented by the minor tangent curve.

Similarly the total birthrate is always less than that which would result from a population and age distribution represented by the major tangent curve.

From this it is clear that, after the original population has once died out, the representative curve can never again contact with the original minor and major tangent curves, but must henceforth be separated from them everywhere by a finite margin (except, of course, where the several curves terminate upon the axis of a).

We may then begin afresh by drawing a new pair of tangent curves, lying within the original pair, and so on indefinitely, until the minor and major tangent curves coincide, and with them also coincides the actual curve of age distribution, which is then of the form

$$N(t)\ c(a) = Ke^{rt}e^{-ra}p(a)$$

which we recognize as the fixed age distribution (3), with

$$N(t) = Ke^{rt}\int_0^\infty e^{-ra}p(a)da$$

In this argument we have expressly excluded the case that the original age distribution curve contacts *continuously* with one of its tangent curves over a range greater than the reproductive period $(a_j - a_i) = A$. If such continuous contact occurs in the 0th generation over a range nA, where $a_j/A > n > 1$, then a simple reflection shows that in the next generation (the first) contact will occur over a range $(n - 1)A$, in the second generation over $(n - 2)A$, etc. A time is therefore reached (in practice very soon), when contact is over a period less than the period of reproduction A. After that the argument set forth above applies.

If the curve representing the original age distribution contacts with one of the tangent curves all the way from $a = 0$ to $a = a_j$, so that $a_j/A \leqq n$, then, of course, the fixed age distribution is practically established *ab initio*, or at any rate from the moment the original population above reproducing age a_j has died out.

It remains to consider the effect of variability in the form of $\beta(a)$ with changes in $c(t,a)$. Some such variability undoubtedly exists owing to the influence of the ages of the male and female constituents of the population upon the frequency of matings. We may, nevertheless, in this case also, define a minor and a major tangent curve (5), (6) in terms of the value of r given by equation (4); in order, however, to make this value determinate, it is now necessary to make some definite disposition regarding the form of $\beta(a)$, which is now variable. Merely for purposes of defining r, we shall suppose that the function $\beta(a)$ under the integral sign has that particular form which corresponds to the fixed age distribution.[5]

We cannot, however, now reason, as before, that the portion of the population represented by the shaded area in fig. 1 will, by itself, reproduce

its own form of age distribution. For its constitution as to sex will in general differ from that of the "normal" population with self-perpetuating age distribution. We must therefore consider three different possibilities:

1. The shaded area alone will produce a population exceeding at all ages the normal or fixed type continuation (8) of the original shaded area. Should this be the case, then the argument presented with regard to the case of invariable $\beta(a)$ holds *a fortiori*, so far as the minor tangent curve is concerned.

2. The shaded area alone will produce a population deficient at some or all ages, as compared with the normal type continuation (8) of the original shaded area. In that case two alternatives present themselves:

a. The deficiency is more than counterbalanced by the additional population produced by that portion of the original population which is represented by the dotted area in fig. 1. In this case also the original argument, so far as the minor tangent curve is concerned, applies essentially as before.

b. After the contributions from all parts of the population have been taken into account, there remains an unbalanced deficiency short of the population defined by (8), in the population resulting from that originally present. In such case the argument presented on the assumption of invariable $\beta(a)$ fails, and the population may move away from, not towards the age distribution (3). Stability of the fixed age distribution may not extend to such displacements as this.

Similar reflections apply, *mutatis mutandis*, as regards the major tangent curve.

If conditions (1) or (2) prevail with respect to the minor, and corresponding conditions with respect to the major tangent curve, then we can argue as in the case of invariable $\beta(a)$, that after expiration of the existing generation a new pair of tangent curves can be drawn, which will lie within the pair defined by (8), (9). And if conditions (1) or (2) still persist with reference to the new tangent curves, the same process of closing in these curves at the expiration of the current generation can be repeated, and so on, as in the argument first presented. Now conditions (1) or (2) will thus continue to prevail for each new pair of tangents, if, as the minor and major tangents close up, $\beta(a)$ approaches a limiting form. In such case, therefore, the age distribution defined by (3), (4) is stable even for displacements of any magnitude, provided always that conditions (1) or (2) prevail, as indicated.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 71.

[2] These circumstances are: (*a*) an invariable life curve (life table); (*b*) an invariable ratio of male to female births; (*c*) an invariable rate per head of procreation at each year of age, for any given sex-and-age distribution in the population.

[3] Lotka, *Amer. J. Sci.*, **26,** 1907 (21); *Ibid.*, **24** (199); *J. Wash. Acad. Sci.*, **3,** 1913 (241, 289).

[4] Sharpe, F. R., and Lotka, A. J., *Phil. Mag.*, April 1911, 435.

[5] A certain ambiguity is introduced here by the fact that, with $\beta(a)$ of variable form, equation (4) might have more than one real root for r. In practice, however, in a human population at any rate, probably only one such root exists, or has any effect upon the course of events.